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On a class of Hamiltonians with (classical) isochronous motions and (quantal) equi-spaced spectra

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Abstract

Hamiltonians *linear* in the momenta and yielding (in the classical context) trajectories *isochronous* in configuration space are considered. It is shown that their motions are *completely periodic* in phase space as well, with the same common period as the orbits in configuration space. Moreover, it is shown that for this particular class of Hamiltonians the semiclassical quantization prescription is *exact*, so that to the *isochronous* character of their *classical* dynamics there corresponds in the *quantized* context an *equidistant* spectrum, for a broad range of ordering prescriptions, including non-symmetrical ones. Examples illustrating these findings are presented.

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1. Introduction

Recently much research has been devoted to the identification and investigation of 'ω-modified' *autonomous* dynamical systems which are *isochronous*, namely which feature an *open* (hence *fully dimensional*) region in their phase space in which *all* their motions are *completely periodic* (namely, periodic in *all* their degrees of freedom) with the *same* fixed period. The 'ω-modified' character of these systems reflects the fact that they are obtained by subjecting unmodified *autonomous* systems to an appropriate change of dependent and independent variables (generally referred to as 'the trick') featuring a *real* (for definiteness, *positive*) parameter ω. This transformation reduces to the identity for $\omega = 0$, but for $\omega > 0$ it generally yields an ω-modified *autonomous* dynamical system which is *isochronous*, typically with the period

$$T = \frac{2\pi}{\omega} \tag{1}$$

or some simple *integer* multiple of T. The generality of this approach is demonstrated by the fairly broad class of autonomous dynamical systems to which it is applicable: the interested reader is referred in this respect to the extended (even if incomplete) list of papers quoted in [1].

These investigations have been mainly (albeit not exclusively) focused on dynamical systems interpretable as (classical, i.e. nonrelativistic and nonquantal) many-body problems characterized by equations of motion of Newtonian type ('acceleration proportional to force'). Of particular interest have been *Hamiltonian* systems, in which these (second-order) Newtonian equations of motion describing the time evolution in configuration space originate from the standard (first-order) Hamiltonian equations describing the time evolution in phase space. However, only in some rather special cases the ω-modified equations of motion obtained in this manner—i.e., yielded by the application of 'the trick' to equations of motion obtained in a *Hamiltonian* context—are themselves *Hamiltonian*, i.e. themselves obtainable starting from an appropriate *Hamiltonian* (here we assume of course that one is not allowed to double the number of canonical variables, since, by doing so, *any* dynamical system can be written in Hamiltonian form: see remark 2.5 below). These special cases are of course particularly interesting, not least because of the possibility of exploring these ω-modified systems also in a *quantal* context, and in particular to test thereby the natural hunch that to the *isochronous* character in the classical context of these Hamiltonian systems there correspond in the quantal context an *equi-spaced* spectrum—at least when the *isochronous* region coincides with the entire (natural) phase space (possibly up to the exclusion of some lower dimensional set), and for some appropriate quantization scheme. In simple cases investigations of this kind have indeed been performed, yielding nontrivial insights on the nature of quantization [2].

A significant development in this context has been the recent introduction [1] of a different kind of trick, applicable to a rather broad class of Hamiltonians, that allows us to generate, from a given time-independent Hamiltonian, an equally time-independent, ω-modified Hamiltonian yielding *isochronous* equations of motion. This has opened the way to the investigation also in the *quantal* context of these ω-modified Hamiltonians, and thereby to illuminate further the relation among the *isochronicity* of classical motions and the (possibly *equi-spaced*) character of the *spectra* of the corresponding quantal Hamiltonian systems. It has also provided the main motivation for obtaining the results reported in the present paper, in which we confine our consideration to *isochronous* Hamiltonians which are *linear* in the canonical momenta (of course analogous results hold for *isochronous* Hamiltonians which are *linear* in the canonical coordinates: indeed in the *classical* context the distinction among canonical momenta and canonical coordinates is purely *semantic*, while in the *quantal* context it corresponds merely to a *Fourier transformation* of the eigenfunctions).

In the following section 2 we introduce these Hamiltonian systems and we discuss their *classical* behaviour. In section 3 we consider these systems in the *quantal* context and we prove the main result of this paper, namely that for this particular class of Hamiltonians the semiclassical quantization prescription is *exact*, so that to the *isochronous* character of their *classical* dynamics there corresponds in the *quantized* context an *equidistant* spectrum; and we show that this actually holds for a broad range of ordering prescriptions, including non-symmetric ones. Examples illustrating these findings are presented in section 4, and some final remarks in section 5.

2. Classical mechanics of isochronous Hamiltonians linear in the momenta

Let us consider the following Hamiltonian:

$$H(\underline{q};\underline{p}) = \underline{a}(\underline{q}) \cdot \underline{p} \equiv \sum_{n=1}^{N} a_n(\underline{q})\, p_n, \tag{2}$$

to which there correspond the Hamiltonian equations of motion

$$\dot{q}_n = \frac{\partial H(\underline{q};\,\underline{p})}{\partial p_n} = a_n(\underline{q}), \tag{3a}$$

$$\dot{p}_n = -\frac{\partial H(\underline{q};\,\underline{p})}{\partial q_n} = -\sum_{m=1}^{N} \frac{\partial a_m(\underline{q})}{\partial q_n} p_m = -\frac{\partial \underline{a}(\underline{q})}{\partial q_n} \cdot \underline{p}, \tag{3b}$$

where the notation is, we trust, self-evident (in particular we hereafter assume indices such as n, m to run from 1 to N, and denote N-vectors by underlined letters).

Remark 2.1. The *linear* character of the Hamiltonian (2) entails that the canonical coordinates evolve according to a set of first-order ODEs, see (3*a*), which is not influenced at all by the corresponding evolution, see (3*b*), of the canonical momenta. Hence this kind of dynamical system should perhaps be called 'Aristotelian' (in contrast to 'Newtonian'), inasmuch as the velocities, rather than the accelerations, equal the 'forces' (namely, the quantities appearing on the right-hand side of the equations of motion, see (3*a*)).

Remark 2.2. Adding to the Hamiltonian (2) a term independent of the canonical momenta, so that it reads

$$H(\underline{q};\,\underline{p}) = b(\underline{q}) + \underline{a}(\underline{q}) \cdot \underline{p}, \tag{4}$$

does not affect at all the dynamics in configuration space (see (3*a*)).

Remark 2.3. Adding to the Hamiltonian (2) a term independent of the canonical momenta of the form $\underline{a}(\underline{q}) \cdot \underline{\nabla} r(\underline{q})$ so that it read

$$H(\underline{q};\,\underline{p}) = \underline{a}(\underline{q}) \cdot \underline{\nabla} r(\underline{q}) + \underline{a}(\underline{q}) \cdot \underline{p} \tag{5a}$$

amounts merely to redefining the momenta according to the rule

$$\underline{p} \mapsto \underline{p} + \underline{\nabla} r(\underline{q}). \tag{5b}$$

Note that this transformation, associated with $\underline{q} \mapsto \underline{q}$, is clearly *canonical*. There is therefore some freedom in adding a momenta-independent term to the Hamiltonian without significantly affecting the momenta dynamics, though it cannot be done in full generality without affecting it.

Remark 2.4. Hamiltonians linear in the momenta are quite special, yet *any* Hamiltonian system can in fact be reformulated as a Hamiltonian system linear in the momenta by *doubling* the number of canonical variables. Indeed let $\check{H}(q_1, \ldots, q_N;\, p_1, \ldots, p_N)$ be a completely general Hamiltonian, yielding the equations of motion

$$\dot{q}_n = \frac{\partial \check{H}}{\partial p_n}, \qquad \dot{p}_n = -\frac{\partial \check{H}}{\partial q_n}, \qquad n = 1, \ldots, N, \tag{6}$$

and define now a new Hamiltonian $\tilde{H}(Q_1, \ldots, Q_{2N};\, P_1, \ldots, P_{2N})$ with double the number of variables, as follows: introduce the $2N$ canonical variables Q_ν, $\nu = 1, \ldots, 2N$, via the assignment

$$Q_n = q_n, \quad Q_{N+n} = p_n, \quad n = 1, \ldots, N; \tag{7}$$

call P_ν the $2N$ canonical momenta conjugated to the $2N$ canonical variables Q_ν, $\nu = 1, \ldots, 2N$, and define the Hamiltonian $\tilde{H}(Q_1, \ldots, Q_{2N}; P_1, \ldots, P_{2N})$ as follows:

$$\tilde{H}(Q_1, \ldots, Q_{2N}; P_1, \ldots, P_{2N}) = \sum_{n=1}^{N} \left[\frac{\partial \check{H}(Q_1, \ldots, Q_N; Q_{N+1}, \ldots, Q_{2N})}{\partial Q_n} P_{N+n} - \frac{\partial \check{H}(Q_1, \ldots, Q_N; Q_{N+1}, \ldots, Q_{2N};)}{\partial Q_{N+n}} P_n \right]. \tag{8}$$

It is then plain that the (first) $2N$ Hamiltonian equations yielded by this Hamiltonian $\tilde{H}(Q_1, \ldots, Q_{2N}; P_1, \ldots, P_{2N})$—which is indeed linear in the momenta!—reproduce the original Hamiltonian equations of motion (6).

The complete equivalence—regarding the time evolution of the N canonical coordinates q_n—among the two Hamiltonian formulations indicated in this remark 2.4 holds of course *only* in the classical context; generally, it does *not* carry over to the quantum case, since the Poisson brackets $\{q_k, \dot{q}_k\}$ vanish for $\tilde{H}$ but not for $\check{H}$, so that quantization will necessarily differ. It is indeed well known (although not always emphasized when teaching quantum mechanics) that *even* Hamiltonians yielding exactly the *same* classical dynamics in configuration space may well yield different quantum dynamics (see for instance [4]).

Also relevant in this connection is the observation that the preceding remark 2.4 is merely a special case of the following more general (if rather trivial)

Remark 2.5. The *most general* dynamical system in configuration space, being characterized by the equations of motions (3*a*), is *Hamiltonian*, being indeed yielded by the Hamiltonian (2).

In the following we limit our consideration to the case in which the dynamics defined by (3*a*) is *isochronous*, that is, we assume there exists a set of initial conditions having *full dimensionality* in phase space for which the orbits in configuration space are all *completely periodic* (i.e., periodic in *all* degrees of freedom) with the *same* period T. Note however that it is not *a priori* certain that the corresponding solutions of the Hamiltonian equations of motion (3) are *completely periodic* in phase space (i.e., that also the time evolution of the canonical momenta is *isochronous*, and with the same period): only if this is so, the corresponding orbits shall close not only in the $\underline{q}$ coordinates, but simultaneously in the $\underline{p}$ coordinates as well.

The first result we now show is that such is, in fact, the case whenever, as we indeed assume, the dynamics given by (3*a*) is *completely periodic*. To prove this we observe first of all that the equations of motion (3*b*) are *linear*, hence we can consider separately the time evolution of the components of the N-vector $\underline{p}$ parallel, respectively orthogonal, to the N-vector $\underline{a}(\underline{q})$. The former coincides clearly (see (2)) with the Hamiltonian divided by the length of the N-vector $\underline{a}(\underline{q})$ and it is therefore certainly *periodic* (since the Hamiltonian is constant throughout the motion, and the vector $\underline{a}(\underline{q})$ is *periodic* inasmuch as $\underline{q}$ itself is periodic: we assume of course that the N-vector function $\underline{a}(\underline{q})$ is *univalent*). To show that the component of the N-vector $\underline{p}$ *perpendicular* to $\underline{a}(\underline{q})$ is also *periodic*, we introduce the scalar function $S(\underline{q})$ defined by the following PDE:

$$\underline{a}(\underline{q}) \cdot \frac{\partial S(\underline{q})}{\partial \underline{q}} \equiv \sum_{n=1}^{N} a_n(\underline{q}) \frac{\partial S(\underline{q})}{\partial q_n} = 0. \tag{9}$$

This is in fact just the Hamilton–Jacobi equation corresponding to the Hamiltonian (2). Note that this *linear* PDE is of *first order*, and can therefore be solved using the method

of *characteristics*. It follows from the general theory of the Hamilton–Jacobi equation that the *characteristics* are in fact determined by the *classical equations of motion*, i.e. by (3*a*). However, we will present the relevant calculation in detail, as the Hamiltonian (2) is a singular one, so that standard theory might not apply.

Let us consider any function $S(\underline{q})$ having the property to be constant along the orbits $\underline{q}(t)$ of (3*a*), that is

$$S[\underline{q}(t)] = \text{const.} \tag{10}$$

Since the orbits are by hypothesis all *periodic*, such functions exist, and they clearly solve (9). We now ask how the components of the *gradient* of $S[\underline{q}(t)]$ vary with time. In component notation one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial S[\underline{q}(t)]}{\partial q_n} = \sum_{m=1}^{N} a_m[\underline{q}(t)]\frac{\partial^2 S[\underline{q}(t)]}{\partial q_n \partial q_m} \tag{11a}$$

$$= -\sum_{m=1}^{N} \frac{\partial a_m[\underline{q}(t)]}{\partial q_n}\frac{\partial S[\underline{q}(t)]}{\partial q_m}, \tag{11b}$$

where the first equality obtains via the equations of motion (3*a*) and the second by taking the gradient of (9). We thus see that the gradient $\partial S/\partial \underline{q}$ of $S(\underline{q})$ satisfies the same evolution equation as the momentum $\underline{p}$ (see (3*b*) and (11*b*)). Therefore, if the initial value of the N-vector $\underline{p}$ of the momenta is perpendicular to (the *initial* value of) $\underline{a}(\underline{q})$, it will evolve just as the gradient of S and therefore return to its original value once the orbit has run through one period in configuration space. Our first result is thus proven.

Let us end this section by also reporting the version of the Hamilton–Jacobi equation at a fixed energy E:

$$\underline{a}(\underline{q}) \cdot \frac{\partial S_E(\underline{q})}{\partial \underline{q}} \equiv \sum_{n=1}^{N} a_n(\underline{q})\frac{\partial S_E(\underline{q})}{\partial q_n} = E. \tag{12}$$

Note that we denote its solution as $S_E(\underline{q})$.

3. Quantum mechanics

Standard quantization of the Hamiltonian (2) yields the operator

$$H = \frac{\hbar}{\mathrm{i}}\sum_{n=1}^{N} a_n(\underline{x})\frac{\partial}{\partial x_n}, \tag{13}$$

and correspondingly the stationary Schrödinger equation (at fixed energy E)

$$\frac{\hbar}{\mathrm{i}}\underline{a}(\underline{x}) \cdot \frac{\partial \psi_E(\underline{x})}{\partial \underline{x}} \equiv \frac{\hbar}{\mathrm{i}}\sum_{n=1}^{N} a_n(\underline{x})\frac{\partial \psi_E(\underline{x})}{\partial x_n} = E\psi_E(\underline{x}). \tag{14}$$

The notation is, we trust, self-evident; but let us emphasize that we ignore for the moment ordering issues. In fact, the quantum Hamiltonian (13) is not even formally symmetric. We shall later show that this does not greatly matter as regards our argument concerning the equi-spaced character of the spectrum; it does, of course, matter when one actually computes the corresponding eigenfunctions of this Hamiltonian (anyway in the examples discussed in section 4 we will always symmetrize the Hamiltonian before quantizing it).

We now state in full the main result of this section.

Theorem 1. *Let the Hamiltonian H be defined as follows:*

$$H = \frac{\hbar}{\mathrm{i}} \left[\sum_{n=1}^{N} a_n(\underline{x}) \frac{\partial}{\partial x_n} + \frac{u}{2} \sum_{n=1}^{N} \left(\frac{\partial a_n}{\partial x_n} \right) \right], \tag{15}$$

where $a_n(\underline{x})$ gives rise to an isochronous system with period T throughout phase space, then for arbitrary values of the parameter u the spectrum of (15) is given by

$$E_k = \frac{2\pi\hbar k}{T} \tag{16}$$

with k an integer.

Remark 3.1. Let us first comment on the significance of the parameter u. The case $u = 1/2$ arises from using the following (more general) quantization prescription in order to obtain a symmetric Hamiltonian:

$$a_n(\underline{q})\, p_n \mapsto \frac{\hbar}{2\mathrm{i}} \left\{ [a_n(\underline{x})]^{\alpha} \frac{\partial}{\partial x_n} [a_n(\underline{x})]^{1-\alpha} + [a_n(\underline{x})]^{1-\alpha} \frac{\partial}{\partial x_n} [a_n(\underline{x})]^{\alpha} \right\} \tag{17}$$

with α an *arbitrary real* constant in the interval $0 \leqslant \alpha \leqslant 1/2$ (for $\alpha = 0$ this correspond to the standard *symmetrization* prescription). It yields for the Hamiltonian operator (2) the following result, independent of the value of α:

$$H = \frac{\hbar}{\mathrm{i}} \sum_{n=1}^{N} a_n(\underline{x}) \frac{\partial}{\partial x_n} + \frac{\hbar}{2\mathrm{i}} \sum_{n=1}^{N} \left[\frac{\partial a_n}{\partial x_n} \right], \tag{18}$$

which corresponds to the quantum Hamiltonian (15) with $u = 1/2$. When $u \neq 1/2$, on the other hand, the Hamiltonian (15) is not symmetric. The theorem also holds in this case, but the eigenvectors belonging to different eigenvalues are then not orthogonal in general.

Proof. We first consider the case in which $u = 0$. We make the following *ansatz* for the wavefunction of the Schrödinger equation which is then given by (14):

$$\psi_E(\underline{x}) = \exp \left[\frac{\mathrm{i} S_E(\underline{x})}{\hbar} \right]. \tag{19}$$

Let us emphasize that this is *not* the point of departure of an approximation technique: we show below that this formula is in fact *exact*. Indeed the insertion of this *ansatz* in (14) yields the relation

$$\underline{a}(\underline{x}) \cdot \frac{\partial S_E(\underline{x})}{\partial \underline{x}} \equiv \sum_{n=1}^{N} a_n(\underline{x}) \frac{\partial S_E(\underline{x})}{\partial x_n} = E, \tag{20}$$

which is just the Hamilton–Jacobi equation at fixed energy (see (12)). We may study this along similar lines as (9). Consider a hypersurface Σ in configuration space of dimension $N - 1$, on which we shall take all initial conditions $\vec{\xi}_0$, where the superimposed arrow denotes here vectors belonging to Σ, as opposed to configuration space vectors, which we continue to denote by underlined letters. Let Σ be chosen so that it never crosses an orbit of (3*a*) more than once. This is achieved as follows: if one considers a reference (closed) orbit starting at $\underline{q}_0 = \vec{\xi}_0$, it is always possible to choose a small tube of orbits around it that has no self intersections (note that at this stage we rely on the assumption that the system is *isochronous* in this region of configuration space with a period T, see below). We may then choose Σ as a section of this closed tube. We may now use the following coordinates for the tube: we consider for any $\underline{x}$ the orbit of (3*a*) which passes through $\underline{x}$ and define $\vec{\xi}_0(\underline{x})$ as the (unique) vector where this orbit crosses Σ and $t(\underline{x})$ as the time required to reach $\underline{x}$ from $\vec{\xi}_0$. The function

$t(\underline{x})$ increases by T (the period of the *isochronous* motion) when $\underline{x}$ returns to Σ after a full round. The function $t(\underline{x})$ is therefore *not* single valued along the orbit: it is rather defined mod T since it increases by T after every turn along the orbit.

It is then clear that a function which only depends on $\vec{\xi}_0$ is *constant* along the orbit. As discussed in the preceding section, it therefore satisfies (9). It then follows via the first Hamiltonian equation (3*a*) (of course with $\underline{q}$ replaced by $\underline{x}$) and (20) that

$$\frac{\mathrm{d}S_E[\underline{x}(t)]}{\mathrm{d}t} \equiv \frac{\mathrm{d}S_E}{\mathrm{d}t} = E. \tag{21}$$

To obtain a solution $S_E(\underline{x})$ of this equation it is then sufficient to set

$$S_E(\underline{x}) = \sigma(\vec{\xi}_0) + Et(\underline{x}). \tag{22}$$

The function σ is arbitrary, but it depends only on the orbit (hence, as indicated by our notation, only on the point $\vec{\xi}_0$ where the orbit intersects the surface Σ), not on the coordinate $\underline{x}$ along the orbit. The function $S_E(\underline{x})$ only depends on $\underline{x}$ via $t(\underline{x})$ (see (22)). It thereby inherits the multivaluedness of $t(\underline{x})$ (as a function of $\underline{x}$) if $E \neq 0$: after each period along the orbit it increases by ET, where T is the period common to all the orbits. Moreover, the *isochronous* property entails that this lack of univalence is identical for all the orbits in the tube: it corresponds for all of them to the same mod T definition of $t(\underline{x})$ along the orbit, with the same T for all the orbits in the tube. But this lack of univalence does *not* affect the wavefunctions $\psi_E(\underline{x})$, now given (see (19)) by the formula

$$\psi_E(\underline{x}) = \exp\left\{\frac{\mathrm{i}[\sigma(\vec{\xi}_0) + Et(\underline{x})]}{\hbar}\right\}, \tag{23}$$

provided the values of the energy E are *quantized* according to the rule (16) stated in the theorem with k *integer*. The condition that the wavefunction be *univalent* is precisely the requirement that characterizes the quantum spectrum, which is thus given just by this formula, (16), with the requirement that k be *integer*. The quantum spectrum is thus indeed *equi-spaced*. Note in particular that there is no zero-point energy: the systems considered are quantized just according to the semiclassical ('Bohr–Sommerfeld') rule.

Let us now proceed to the general case, in which $u \neq 0$. If we now use again the *ansatz* (19) to study the corresponding Schrödinger equation, we obtain again an equation of Hamilton–Jacobi type (at fixed energy):

$$\sum_{n=1}^{N} a_n(\underline{x})\frac{\partial S_E}{\partial x_n} + \frac{\hbar}{2\mathrm{i}}\sum_{n=1}^{N}\frac{\partial a_n}{\partial x_n} = E. \tag{24}$$

This can again be solved by the method of characteristics, and the characteristics are again given just by the solutions of (3*a*), since the additional term caused by the new ordering is independent of $\underline{p}$ and therefore does not influence the dynamics in configuration space (see remark 2.2). Hence if we introduce the coordinates $(t, \vec{\xi}_0)$ which we used above, we find from (24)

$$\frac{\mathrm{d}S_E}{\mathrm{d}t} = E - \frac{\hbar}{2\mathrm{i}}\sum_{n=1}^{N}\frac{\partial a_n[\underline{x}(t)]}{\partial x_n}, \tag{25}$$

to be compared with (21). In order to understand that the additional term due to the ordering prescription is irrelevant to quantization in this particular case, it is therefore sufficient to show that

$$\int_0^T \mathrm{d}t \sum_{n=1}^{N}\frac{\partial a_n[\underline{x}(t)]}{\partial x_n} = 0. \tag{26}$$

But from (3*b*) we infer that

$$\sum_{n=1}^{N} \frac{\partial a_n[\underline{x}(t)]}{\partial x_n} = \frac{\mathrm{d}}{\mathrm{d}t} \ln \left\{ \frac{\partial[\underline{p}(t)]}{\partial[\underline{p}(0)]} \right\}, \tag{27}$$

where we use the notation

$$\frac{\partial(\underline{y})}{\partial(\underline{x})} \equiv \det \left[\frac{\partial y_n(\underline{x})}{\partial x_m} \right] \tag{28}$$

to denote the Jacobian of the N coordinates $\underline{y}$ with respect to the N coordinates $\underline{x}$. And from (27) and the periodicity of the momenta $p_n(t)$ shown in the preceding section, there follows that the integral in the left-hand side of (26) indeed vanishes. □

While this concludes our argument, up to the additional considerations given below, there is one important *caveat* that must be issued here. We assumed in the above argument that the choice of initial conditions for the trajectories could be made without any limitation. However if, in the model under consideration, there are certain initial conditions that are forbidden because they correspond to singularities of the forces, then a condition sufficient (albeit possibly not necessary) for the validity of our conclusion is that the configuration space region where such singular values occur be excluded from consideration—provided this can be done in the quantum context. This fact will be illustrated in the examples provided in the next section 4, and in fact it provides the main motivation for the detailed and extended character of that section, in particular the selection and plurality of the examples presented there.

Finally, let us consider a somewhat more general situation: indeed, it often happens that the configuration space separates into various subsets Ω_j such that the solutions of the classical equations of motion are *completely periodic* with *different* periods T_j in each Ω_j. There may further exist an $\bar{\Omega}$ in which the motion is *multiply periodic*, or *not periodic* at all. The fundamental result for such systems can be phrased as follows: within each Ω_j, it is clear that the whole construction described above can be carried out with no change except that of replacing T by T_j. Thus to each Ω_j there corresponds an *equidistant* spectrum with level spacing h/T_j. What we wish to show here is the following: that every function which has its support in Ω_j can be generated as a linear combination of the eigenfunctions of H obtained via the above construction. It is enough to show that every function inside a tube around some reference periodic orbit can be so generated. Let us again introduce the transverse hypersurface Σ and the coordinates $\vec{\xi}(\underline{x})$ and $t(\underline{x})$. The eigenfunctions of H are then of the form

$$\psi_E(\underline{x}) = \Phi[\vec{\xi}(\underline{x})] \exp\left[\frac{\mathrm{i}Et(\underline{x})}{\hbar} \right], \tag{29}$$

where $\Phi(\vec{\xi})$ is an arbitrary function on Σ vanishing outside the tube. Since E takes as values all integer multiples of h/T_j (and only these), it follows from Fourier analysis that all functions of the form $\Phi[\vec{\xi}(\underline{x})]\ P_j[t(\underline{x})]$ with $P_j(t)$ any *periodic* function of t having period T_j can be obtained as linear combinations of the eigenfunctions $\psi_E(\underline{x})$. But any function of $\underline{x}$ can be expressed as

$$\sum_{\ell=0}^{\infty} \Phi_\ell[\vec{\xi}(\underline{x})]\ P_\ell[t(\underline{x})], \tag{30}$$

hence the result is proved. The eigenfunctions obtained for Ω_j through our construction therefore provide a complete eigenbasis for the Hilbert space $L^2(\Omega_j)$ of square integrable

functions with support in Ω_j. It therefore follows that one can perform the above construction independently for all the sectors Ω_j and obtain the spectrum corresponding to the union of all the Ω_j having *isochronous* behaviour. The situation in the region of configuration space $\bar{\Omega}$ (if it exists) remains, of course, unspecified, but we may state that the complications (both of the classical motions and of the quantum spectrum associated with eigenfunctions having their support in this region) do not affect that part of Hilbert space corresponding to functions having instead their support outside $\bar{\Omega}$.

4. Examples

In this section we illustrate our findings via some examples.

4.1. A very elementary example

We begin with a very elementary example to serve as a warm-up, and also to establish some (standard) notation that will be used also in subsequent examples.

Consider the following Hamiltonian describing a rotator in the (horizontal) plane:

$$H(x, y; p_x, p_y) = \omega(xp_y - yp_x) = \omega[\hat{z} \cdot (\vec{r} \wedge \vec{p})] = \omega p_\theta. \tag{31}$$

Notation: ω is a positive constant; x and y are the Cartesian coordinates of the moving point in the horizontal plane and p_x, p_y the corresponding canonical momenta; $\hat{z} \equiv (0, 0, 1)$ is the *three-dimensional* unit vector orthogonal to the horizontal plane and $\vec{r} \equiv (x, y, 0)$, $\vec{p} \equiv (p_x, p_y, 0)$ are correspondingly a convenient way to express the canonical coordinates in a *three-dimensional* context (allowing a covariant description, see the second expression on the right-hand side of (31), where the dot respectively the wedge symbols sandwiched among two 3-vectors denote of course the scalar respectively vector products of these two 3-vectors); finally r, θ are the standard circular coordinates in the plane,

$$x = r\cos(\theta), \qquad y = r\sin(\theta); \qquad r^2 = x^2 + y^2, \qquad \tan(\theta) = \frac{y}{x}, \tag{32}$$

and p_r, p_θ the corresponding canonical momenta,

$$p_r = p_x\cos(\theta) + p_y\sin(\theta), \qquad p_\theta = r[-p_x\sin(\theta) + p_y\cos(\theta)], \tag{33a}$$

$$p_x = p_r\cos(\theta) - \frac{p_\theta\sin(\theta)}{r}, \qquad p_y = p_r\sin(\theta) + \frac{p_\theta\cos(\theta)}{r}. \tag{33b}$$

The equations of motion yielded by the Hamiltonian (31) are most simply expressed in circular coordinates, indeed in self-evident notation they read

$$\dot{\theta} = \omega, \quad \dot{r} = 0; \quad \dot{p}_r = \dot{p}_\theta = 0, \tag{34}$$

hence their solution is

$$r(t) = r(0), \tag{35a}$$

$$\theta(t) = \theta(0) + \omega t, \tag{35b}$$

describing circular motions in the plane *all* periodic with the same period T, see (1), consistently with the formula

$$\theta(t + T) = \theta(t) + 2\pi. \tag{36}$$

The corresponding quantum problem, the formulation of which is again particularly simple in circular coordinates, is characterized by the Hamiltonian operator

$$H = \frac{\hbar\omega}{\mathrm{i}}\frac{\partial}{\partial\theta} \tag{37}$$

(see the last expression on the right-hand side of (31), and the relations among the Cartesian and circular coordinates (32) entailing that to the standard quantization prescription in Cartesian coordinates, $p_x \mapsto -\hbar \mathrm{i}\partial/\partial x$, $p_y \mapsto -\hbar \mathrm{i}\partial/\partial y$ there corresponds the analogous prescription $p_r \mapsto -\hbar \mathrm{i}\partial/\partial r$, $p_\theta \mapsto -\hbar \mathrm{i}\partial/\partial\theta$). Hence the corresponding stationary Schrödinger equation (at energy E) reads

$$\frac{\hbar\omega}{\mathrm{i}}\frac{\partial\psi_E(r,\theta)}{\partial\theta} = E\psi_E(r,\theta). \tag{38}$$

The solution of this equation is

$$\psi_E(r,\theta) = \psi(r,\theta_0)\exp\left[\frac{\mathrm{i}E(\theta-\theta_0)}{\hbar\omega}\right], \tag{39}$$

where $\psi(r,\vartheta_0)$ is an *arbitrary* function of r, and the requirement that $\psi_E(r,\theta)$ be *univalent* in the plane,

$$\psi_E(r,\theta+2\pi) = \psi_E(r,\theta), \tag{40}$$

entails that the energy E is restricted to belong to the following spectrum of the operator H:

$$E_k = \hbar\omega k \tag{41}$$

with k *integer*. Note that each of this discrete energy eigenvalues is *infinitely degenerate*, due to the arbitrariness of the r-dependence of the corresponding eigenfunction, see (39). This arbitrariness corresponds to the fact that the radius of the circular orbit in the classical case is neither fixed by the dynamics nor by the value of the energy of the motion, but merely by the initial data.

Remark 4.1. The spectrum (41) has been yielded by the requirement that the eigenfunctions of the Hamiltonian be *univalent*. The additional requirement that the eigenfunctions be *normalizable* can be trivially met in this case, given the arbitrariness of their r-dependence.

Remark 4.2. This example raises no ordering problem.

The connection with the treatment given above (and with the corresponding notation) is as follows: in Cartesian coordinates $\underline{a}$ is the two-vector

$$\underline{a}(\underline{q}) \equiv \underline{a}(x,y) = (-\omega y, \omega x), \tag{42a}$$

(see (2) and the equations that follow it) and in circular coordinates

$$\underline{a}(\underline{q}) \equiv \underline{a}(r,\theta) = (0,\omega). \tag{42b}$$

In the classical context, the Hamilton–Jacobi equation, see (9), reads, in circular coordinates,

$$\omega\frac{\partial S(r,\theta)}{\partial\theta} = 0, \tag{43a}$$

hence

$$S(r,\theta) = S(r,\theta_0), \tag{43b}$$

where θ_0 is an arbitrarily assigned fixed value of θ and $S(r,\theta_0)$ is to be set by the initial data. This function is indeed constant (see (10)) along the (circular) trajectories, see (35). Likewise, the Hamilton–Jacobi equation at fixed energy, see (12), reads

$$\omega\frac{\partial S_E(r,\theta)}{\partial\theta} = E, \tag{44}$$

hence the function $S_E(r,\theta)$ that satisfies this equation reads

$$S_E(r,\theta) = S_E(r,\theta_0) + \frac{E(\theta-\theta_0)}{\omega}, \tag{45}$$

and via the Hamiltonian equations (34) one immediately verifies the validity of (21). Moreover from the formulae of the classical trajectories (35) we see that this equation entails the relation

$$S_E(r,\theta) = \sigma(r) + Et, \tag{46a}$$

hence

$$t(r,\theta) = \frac{S_E(r,\theta) - \sigma(r)}{E} \tag{46b}$$

where we set

$$\sigma(r) = S_E(r,\theta_0) + \frac{E[\theta(0) - \theta_0]}{\omega}. \tag{46c}$$

The dropping of the label E from this function $\sigma(r)$ is justified inasmuch as this is an essentially arbitrary function.

Turning now to quantum mechanics we see that the *ansatz* (19) yields, via (45), precisely the Schrödinger eigenfunction $\psi_E(r,\theta)$, see (39), which can also be written, via (46), in the guise

$$\psi_E(r,\theta) = \psi(r)\exp\left[\frac{\mathrm{i}Et(r,\theta)}{\hbar}\right], \tag{47}$$

with $\psi(r)$ an essentially arbitrary function. This shows—congruently with the treatment given above—that the quantization condition (41) is related with the *periodicity* of the classical trajectories, entailing the relation

$$t(r,\theta+2\pi) = t(r,\theta) + T = t(r,\theta) + \frac{2\pi}{\omega}, \tag{48}$$

that is clearly congruent with (35*b*), (45), (46*b*), (36) and (1).

4.2. *Another simple example*

The next example we consider is characterized by the Hamiltonian (in circular coordinates)

$$H(r,\theta;\, p_r,\, p_\theta) = a(r,\theta)\,p_\theta, \tag{49a}$$

$$a(r,\theta) = \omega\left(\frac{R}{r}\right)\frac{1 + 2\left(\frac{r}{R}\right)\cos\theta + \left(\frac{r}{R}\right)^2}{\left(\frac{r}{R}\right) + \cos\theta} \tag{49b}$$

$$= \omega\left(\frac{R}{r}\right)\frac{\left[1 + \left(\frac{r}{R}\right)\exp(\mathrm{i}\theta)\right]\left[1 + \left(\frac{r}{R}\right)\exp(-\mathrm{i}\theta)\right]}{\left(\frac{r}{R}\right) + \cos\theta}, \tag{49c}$$

entailing, in the classical context, the equations of motion

$$\dot\theta = a(r,\theta), \quad \dot r = 0. \tag{49d}$$

Here ω and R are two *positive* constants. Note that the right-hand side of the (first) equation of motion (49*d*) becomes singular at $r = 0$, and moreover, if $r \leqslant R$, at $\cos\theta = -r/R$.

A standard computation yields, in the *classical* context, the following solution for the initial-value problem for these equations of motion:

$$\frac{1 + \left(\frac{r}{R}\right)\exp(-\mathrm{i}\theta)}{1 + \left(\frac{r}{R}\right)\exp(\mathrm{i}\theta)} = \exp(-2\mathrm{i}\omega t), \tag{50}$$

where, without loss of generality, we set $t = 0$ when $\theta = 0$ (and of course $r = r(0)$). Because the right-hand side of this equation is clearly periodic with primitive period $T/2 = \pi/\omega$, one

might infer that the same conclusion is implied by this formula for the time evolution of $\theta(t)$. This is, however, not the case. Indeed by solving this equation for $\exp(\mathrm{i}\theta)$ one gets

$$\mathrm{e}^{\mathrm{i}\theta} = \frac{R}{2r}\left\{\mathrm{e}^{2\mathrm{i}\omega t} - 1 + \left[\mathrm{e}^{2\mathrm{i}\omega t} - \rho_+\left(\frac{r}{R}\right)\right]^{1/2}\left[\mathrm{e}^{2\mathrm{i}\omega t} - \rho_-\left(\frac{r}{R}\right)\right]^{1/2}\right\}, \tag{51a}$$

with

$$\rho_\pm(x) = 1 - 2x^2 \pm 2x(x^2-1)^{1/2}. \tag{51b}$$

It is easily seen that the last formula implies that, for $x = 1$, $\rho_\pm(x) = -1$; for $x > 1$, $\rho_\pm(x)$ are both *real* and $-1 < \rho_+(x) < 0$ hence $|\rho_+(x)| < 1$ while $\rho_-(x) < -1$ hence $|\rho_-(x)| > 1$; for $x < 1$, $\rho_\pm(x)$ are both *complex* and $|\rho_\pm(x)| = 1$. There are therefore three different regimes, corresponding to the following three assignments for r (that is of course fixed by the initial condition, $r = r(0)$, with the obvious exclusion of $r(0) = 0$):

(i) $r = R$, implying $\theta(t) = 2\omega t$, entailing a circular uniform motion periodic with primitive period $T/2 = \pi/\omega$.
(ii) $r > R$, entailing a circular non uniform motion periodic with primitive period $T = 2\pi/\omega$.
(iii) $r < R$, entailing that, at some finite time t_s (smaller than $T/2 = \pi/\omega$) the trajectory hits a singularity: it is indeed easily seen that, at that time t_s, $\cos\theta(t_s) = -r/R$, hence the right-hand side of the (first) equation of motion (49*d*) blows up.

Let us now turn to the quantum case. The corresponding Hamiltonian operator reads

$$H = \frac{\hbar}{\mathrm{i}}\left\{a(r,\theta)\frac{\partial}{\partial\theta} + \frac{u}{2}\left[\frac{\partial a(r,\theta)}{\partial\theta}\right]\right\}, \tag{52}$$

where the second term in the right-hand side has been introduced so as to make this Hamiltonian operator symmetrical: hence (see (18)) the parameter u should be assigned the value $u = 1$, but we introduced it here as a free parameter in order to be able to trace the effect of the Hamiltonian symmetrization (see below). Here of course $a(r,\theta)$ is defined by (49*b*) or equivalently (49*c*) (one or the other of these two equivalent versions is more convenient to check some of the following computations).

The corresponding stationary Schrödinger equation reads

$$\frac{\hbar}{\mathrm{i}}\left\{a(r,\theta)\frac{\partial}{\partial\theta} + \frac{u}{2}\left[\frac{\partial a(r,\theta)}{\partial\theta}\right]\right\}\psi_E(r,\theta) = E\psi_E(r,\theta), \tag{53}$$

and it is easy to verify that the *general* solution of this linear PDE reads as follows (up to a constant multiplicative factor, that can be adjusted to normalize this eigenfunction):

$$\psi_E(r,\theta) = [a(r,\theta)]^{-u/2} f(r)[\varphi(r,\theta)]^{k/2}, \tag{54a}$$

with $a(r,\theta)$ defined of course as above (see (49*b*) or (49*c*)), $f(r)$ an *arbitrary* function,

$$\varphi(r,\theta) = \frac{1 + \left(\frac{r}{R}\right)\exp(\mathrm{i}\theta)}{1 + \left(\frac{r}{R}\right)\exp(-\mathrm{i}\theta)} \tag{54b}$$

and the 'energy eigenvalue' E related to the constant k by the formula

$$E = \hbar\omega k. \tag{55}$$

Note that for real r and θ (as here considered) the (*complex*) function $\varphi(r,\theta)$ has unit modulus, $|\varphi(r,\theta)| = 1$ (hence it does not diverge), and that it is of course *univalent* in the plane. Also note that the function $[\varphi(r,\theta)]^{k/2}$ is clearly also *univalent* in the plane if $r < R$ even if the constant k is not an *integer*, while if $r > R$ it is *univalent* in the plane only if the constant k is an *integer*.

The requirement that the eigenfunction $\psi_E(r,\theta)$ with $u = 1$ (as implied by the symmetrization of the Hamiltonian) be normalizable (i.e. that the integral over the entire plane of its modulus squared be *finite*) clearly requires that the *a priori arbitrary* function $f(r)$ vanish for $0 \leqslant r \leqslant R$ and that it also vanish (faster than the inverse of r) as $r \to \infty$. Hence this function $f(r)$ cannot vanish identically for $R < r < \infty$, and as a consequence the requirement that the eigenfunction $\psi_E(r,\theta)$ be *univalent* in the plane entails that the constant k must be an *integer*. Hence we conclude that the energy spectrum is *equi-spaced*, see (55). Each eigenvalue is then infinitely degenerate, due to the *arbitrariness* of the function $f(r)$, which is only required to vanish for $0 \leqslant r \leqslant R$ and as $r \to \infty$.

Note that this conclusion would not have been obtained if one had not required the Hamiltonian to be symmetrized. Indeed if in the expression (54*a*) of the eigenfunction the constant u were set to zero, one could well choose a set of eigenfunctions characterized by a function $f(r)$ vanishing for $r \geqslant R$, and then the requirement that the eigenfunction be *univalent* would *not* impose that the constant k be *integer*. One would then be led to conclude that the energy spectrum also has a continuous component. Note however that this does not contradict our previous argument that the spectrum is independent of ordering issues, and in particular that the same equi-spaced spectrum is obtained even for a non-symmetrized Hamiltonian. Indeed this argument only applied to Hamiltonians yielding *isochronous* motions in the entire phase space: if the Hamiltonian is not *isochronous* in the entire (natural) phase space, then the result only holds in those parts of phase space for which the classical dynamics is indeed *isochronous*. It is therefore clear that our results do not apply to that part of phase space for which the classical trajectories run into a singularity at a finite time, since then the system is clearly *not isochronous*. On the other hand, for wavefunctions localized in the region $r > R$ where the system does behave *isochronously*, the spectrum obtained is in fact equi-spaced, independently of the ordering prescription, just as stated above.

4.3. A not-so-trivial example

The third example we consider is again a one-body problem in the plane and it is characterized by the *isochronous* Hamiltonian (in circular coordinates)

$$H(r,\theta;p_r,p_\theta) = -\frac{\omega}{\gamma}\left\{\left[1-\left(\frac{r}{R}\right)^{-\gamma}\sin(\gamma\theta)\right]p_\theta + \left(\frac{r}{R}\right)^{-\gamma}\cos(\gamma\theta)rp_r\right\}, \tag{56}$$

where ω and R are two *arbitrary positive* constants. In order that this Hamiltonian be *univalent* in the entire plane the parameter γ must be *integer*: H is then *periodic* in θ with period $2\pi/\gamma$ hence also with period 2π. In the case with *real* but *not integer* γ the problem could be studied in the sector of the plane characterized by the restriction $0 \leqslant \theta \leqslant 2\pi/\gamma$, but such a model, and its quantized version, are sufficiently interesting to warrant a separate treatment. So hereafter we restrict consideration to *integer* values of the parameter γ. And we of course also assume throughout that the number γ does *not* vanish, see (56); in fact, to streamline our presentation we assume hereafter that

$$|\gamma| \geqslant 2, \quad \gamma = \pm 2 \quad \text{or} \quad \pm 3 \quad \text{or} \quad \pm 4, \ldots, \tag{57}$$

to avoid the need to treat the somewhat special, and not particularly interesting, case with $|\gamma| = 1$.

Let us deal firstly with the *classical* version of this problem. In this context the (configuration space part of the) Hamiltonian equations of motion read

$$\frac{\dot r}{r} = -\frac{\omega}{\gamma}\left(\frac{r}{R}\right)^{-\gamma}\cos(\gamma\theta), \tag{58a}$$

$$\dot\theta = -\frac{\omega}{\gamma}\left[1-\left(\frac{r}{R}\right)^{-\gamma}\sin(\gamma\theta)\right], \tag{58b}$$

and their solution is provided by the formula

$$r(t)\exp[\mathrm{i}\theta(t)] = [r(0)]\exp[\mathrm{i}\theta(0)]\exp\left(-\frac{\mathrm{i}\omega t}{\gamma}\right)\cdot\left\{1-\left[\frac{r(0)\exp\mathrm{i}\theta(0)}{R}\right]^{-\gamma}\frac{\exp(\mathrm{i}\omega t)-1}{\mathrm{i}}\right\}^{1/\gamma} \tag{59a}$$

$$= R\mathrm{i}^{1/\gamma}[1-\beta\exp(-\mathrm{i}\omega t)]^{1/\gamma} \tag{59b}$$

$$= R\left(\frac{\beta}{\mathrm{i}}\right)^{1/\gamma}\exp\left(-\frac{\mathrm{i}\omega t}{\gamma}\right)\left[1-\frac{\exp(\mathrm{i}\omega t)}{\beta}\right]^{1/\gamma}. \tag{59c}$$

(Before proceeding further, let us mention that the reader having difficulty to verify that this formula (59) does indeed provide the solution to the initial-value problem for the Hamiltonian equations of motion (58), and/or being baffled by the way we managed to find this solution, and indeed to invent the *isochronous* Hamiltonian (56), will find an explanation in appendix A; we relegate this explanation there not to interrupt the flow of the argument here).

The second and third versions of this formula (59), which are clearly equivalent to the first provided

$$\beta = 1+\mathrm{i}\left[\frac{r(0)}{R}\right]^{\gamma}\exp[\mathrm{i}\gamma\theta(0)], \tag{59d}$$

have been displayed to make it evident that this solution is *periodic* with primitive period T (see (1)) whenever the initial data entail $|\beta|<1$, and with primitive period γT whenever the initial data entail $|\beta|>1$, thereby confirming the *isochronous* character of this model. The two sets of initial data corresponding to $|\beta|<1$ respectively $|\beta|>1$ are separated by those special initial data such that $|\beta|=1$, for which the solution has a branch point occurring at the (*real*) time t_s defined mod T by the relation $\exp(\mathrm{i}\omega t_s)=\beta$, at which time $r(t)$ diverges or vanishes depending whether the integer γ is *negative* or *positive*. Let us call C the contour in the plane defined by (initial) points such that $|\beta|=1$. It is easily seen that it is characterized by the equation

$$\left(\frac{r_0}{R}\right)^{\gamma} = 2\sin(\gamma\theta_0), \tag{60}$$

where we denote as r_0,θ_0 the circular coordinates along this curve (and as x_0, y_0 the corresponding Cartesian coordinates, see below). For $\gamma>1$ the curve C is described as follows: it consists of γ copies of a basic unit which is the curve described by a point that starts out horizontally and returns to the origin at an angle π/γ after making a closed path inside the sector $0\leqslant\theta\leqslant\pi/\gamma$. This basic unit, or lobe, is then rotated γ times by an angle $2\pi/\gamma$ to yield the full curve C. For $\gamma<-1$, the curve C will be the result of mapping the curve for $|\gamma|$, which is as described above, by the transformation

$$\tilde r = \frac{R^2}{r}. \tag{61}$$

It therefore follows that for $\gamma<-1$ each lobe of C goes to infinity along the appropriate angles, instead of going to zero. Figure 1 provides illustrations of both cases.

Let us now also discuss briefly the equilibria of this dynamical system, since this turns out to be relevant to understand some aspects of the corresponding quantum system, see below. From formulae (59) one easily sees that this system has $|\gamma|$ equilibrium configurations characterized by the following values of the circular coordinates (see (58)):

$$r_j = R, \qquad \theta_j = \frac{2\pi}{\gamma}\left(j+\frac{1}{4}\right). \tag{62}$$

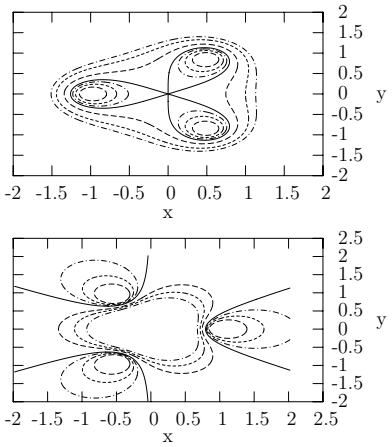

Figure 1. Level sets of $\chi(r,\theta)$, see (65), for $\gamma = 3$ (upper figure) and $\gamma = -3$ (lower figure). The curves correspond to the values $4, 2, 1, 0$ (corresponding to C in the text), -0.2, -0.4 and -0.6. These level curves correspond to orbits in configuration space. The division in four classes is apparent: one in each lobe of C and another one surrounding either C or the origin. Note that C is given as a continuous line, whereas the other curves are not.

We denote these equilibrium configurations by η_j. Here j is an *integer* in the interval $1 \leqslant j \leqslant |\gamma|$. Hence, for $\gamma > 1$, each lobe of C contains exactly one of the equilibria defined in (62). For $\gamma < -1$, the curve C and the equilibria arise from the corresponding values for $|\gamma|$ through the transformation given in (61). For $\gamma < -1$, the origin ($r = 0$) is an additional equilibrium position, whereas, for $\gamma > 1$, infinity ($r = \infty$) is an (improper) equilibrium (note that this is not true for $\gamma = 1$; this motivates our exclusion of this case, in order to simplify our presentation). We note that since the system is two dimensional and has only periodic orbits, *all* orbits necessarily rotate around one, or possibly several, equilibria, as illustrated in figure 1.

It is not difficult to obtain explicit expressions for $r(t)$ and $\theta(t)$ from the solution formula (59), but we leave their display as an exercise for the diligent reader. To understand the qualitative nature of the motions—which is quite different depending whether the modulus of β is less or larger than unity—it is convenient (and sufficient) to consider two extremal cases when the value of this quantity is either quite small or quite large. In the first case, $|\beta| \ll 1$, by expanding in β formula (59*b*) one easily gets (see (59*d*))

$$r(t) = R\left[1 - \frac{|\beta|}{\gamma}\cos(\varphi - \omega t)\right] + O(\beta^2), \tag{63a}$$

$$\theta(t) = -\frac{|\beta|}{\gamma}\sin(\varphi - \omega t) + O(\beta^2); \tag{63b}$$

in the second case, $|\beta| \gg 1$, by expanding in $1/\beta$ formula (59*c*) one gets instead

$$r(t) = R|\beta|^{1/\gamma}\left[1 - \frac{1}{\gamma|\beta|}\cos(\varphi - \omega t)\right] + O(\beta^2), \tag{64a}$$

$$\theta(t) = -\frac{\omega}{\gamma}t \pm \frac{\pi}{\gamma} - \frac{1}{\gamma|\beta|}\sin(\varphi - \omega t) + O(\beta^{-2}) \bmod 2\pi. \tag{64b}$$

The qualitative behaviour of the trajectories can now be inferred from these expressions. In the case with *positive* γ, $\gamma > 1$, hence when the curve C is *closed* (see above), if $|\beta| < 1$ the

motion takes place (with period T, see also above) around one of the finite equilibria η_j and inside C, while if $|\beta| > 1$ the motion takes place (with period $|\gamma|\,T$, see also above) outside C, with the moving particle in fact rotating around C and hence around all finite equilibria, or (in some sense equivalently) around the equilibrium at infinity; and in the special case with $|\beta| = 1$ the motion goes *through* the origin. If instead γ is *negative*, $\gamma < -1$, hence the curve C is *open* (see above), if $|\beta| < 1$ the motion takes again place (with period T, see also above) around one of the non-zero equilibria η_j, while if $|\beta| > 1$ the motion takes place (with period $|\gamma|\,T$, see also above) moving around the origin, which, as stated above, is also an equilibrium point in this case. As $|\beta| \to 1$, the trajectories become larger and larger, independently of the side of C on which the motion takes place. Note that the trajectories cannot in any case (except for the special initial data such that $|\beta| = 1$) cross the curve C. Again, figure 1 illustrates what happens.

We may therefore divide orbits into classes according to the equilibrium around which they turn. As we have seen, they either turn around exactly one of the equilibrium configurations η_j defined in (62) or they turn around the origin if $\gamma < -1$, and around C if $\gamma > 1$. We therefore say an orbit is of class j if it orbits around the equilibrium configuration η_j, with $1 \leqslant j \leqslant |\gamma|$, and of class *zero* otherwise. We further define the integer-valued function $\kappa(r, \theta)$ as the class of the orbit going through (r, θ). This function is defined everywhere except on C.

It also turns out to be convenient to introduce the function

$$\chi(r, \theta) = \left(\frac{r}{R}\right)^{\gamma} \left[\left(\frac{r}{R}\right)^{\gamma} - 2\sin(\gamma\theta)\right]. \tag{65}$$

It is readily found that the (*classical*) Poisson bracket of this function of the canonical coordinates with the Hamiltonian (56) vanishes,

$$\{H, \chi\} = \frac{\partial H(r, \theta; p_r, p_\theta)}{\partial p_r} \frac{\partial \chi(r, \theta)}{\partial r} + \frac{\partial H(r, \theta; p_r, p_\theta)}{\partial p_\theta} \frac{\partial \chi(r, \theta)}{\partial \theta} = 0, \tag{66}$$

implying that $\chi(r, \theta)$ is a conserved quantity (depending only on the canonical coordinates) of our Hamiltonian system. The constancy of $\chi(r, \theta)$ through the *classical* time evolution implies that it can be used to identify the classical orbits. In fact, the two quantities $\chi(r, \theta)$ and $\kappa(r, \theta)$ are both conserved and their values characterize the orbit going through the point (r, θ) *uniquely*. An illustration of these definitions is found in figure 1: there we show the level curves of $\chi(r, \theta)$ for $\gamma = 3$ and $\gamma = -3$. The level curves correspond to classical orbits except for the level curve corresponding to the vanishing of $\chi(r, \theta)$, which is the singular curve C. The three lobes of C are readily identified, as are the four classes of orbits.

Finally, let us note that the solution formula (59) implies that the time taken to go along an orbit from the initial values $r(0), \theta(0)$ to the values r, θ is given by the formula

$$\mathcal{T}[r, \theta; r(0), \theta(0)] = \frac{\mathrm{i}}{\omega} \log\left[\frac{1 + \mathrm{i}\left(\frac{r}{R}\right)^{\gamma} \exp(\mathrm{i}\gamma\theta)}{\beta}\right] \tag{67a}$$

$$= \frac{\mathrm{i}}{\omega} \log\left\{\frac{1 + \mathrm{i}\left(\frac{r}{R}\right)^{\gamma} \exp(\mathrm{i}\gamma\theta)}{1 + \mathrm{i}\left[\frac{r(0)}{R}\right]^{\gamma} \exp[\mathrm{i}\gamma\theta(0)]}\right\}. \tag{67b}$$

Note that here we have emphasized explicitly the dependence on the four variables $r, \theta, r(0), \theta(0)$, but these variables are *not* independent, since they must lie on the same trajectory of the system. The condition for this to hold is most simply expressed by requiring that $\mathcal{T}[r, \theta; r(0), \theta(0)]$ be a *real* quantity, namely that the argument of the logarithm have unit modulus. Let us now consider the quantum version of this problem. First of all one should symmetrize the Hamiltonian (56) (see (17) with $\alpha = 0$), so that the corresponding quantum

operator becomes (in circular coordinates)

$$H = -\frac{\hbar\omega}{\mathrm{i}\gamma}\left\{\left[1-\left(\frac{r}{R}\right)^{-\gamma}\sin(\gamma\theta)\right]\frac{\partial}{\partial\theta}+\left(\frac{r}{R}\right)^{-\gamma}\cos(\gamma\theta)r\frac{\partial}{\partial r}+\frac{2\gamma-1}{2}\left(\frac{r}{R}\right)^{-\gamma}\cos(\gamma\theta)\right\}, \tag{68}$$

and correspondingly the stationary Schrödinger equation (in circular coordinates) reads

$$-\frac{\hbar\omega}{\mathrm{i}\gamma}\left\{\left[1-\left(\frac{r}{R}\right)^{-\gamma}\sin(\gamma\theta)\right]\frac{\partial}{\partial\theta}+\left(\frac{r}{R}\right)^{-\gamma}\cos(\gamma\theta)r\frac{\partial}{\partial r}+\frac{2\gamma-1}{2}\left(\frac{r}{R}\right)^{-\gamma}\cos(\gamma\theta)\right\}\psi_E(r,\theta)=E\psi_E(r,\theta). \tag{69}$$

It is now easy to verify that the solution of this linear PDE reads as follows:

$$\psi_k(r,\theta)=r^{(2\gamma-1)/2}f[\chi(r,\theta),\kappa(r,\theta)][\varphi(r,\theta)]^{k/2}, \tag{70a}$$

$$\varphi(r,\theta)=\frac{1-\mathrm{i}\left(\frac{r}{R}\right)^{\gamma}\exp(-\mathrm{i}\gamma\theta)}{1+\mathrm{i}\left(\frac{r}{R}\right)^{\gamma}\exp(\mathrm{i}\gamma\theta)}, \tag{70b}$$

where $\chi(r,\theta)$ is defined by (65) and the 'energy eigenvalue' E is related to the constant k by the formula

$$E=E_k=\hbar\omega k. \tag{71}$$

Note that the function $\varphi(r,\theta)$, see (70*b*), has unit modulus, $|\varphi(r,\theta)|=1$, while the constant k, and the function $f(\chi,\kappa)$, are *a priori arbitrary*. It turns out, as we shall see later, that (70) provides the *general* solution of the stationary Schrödinger equation (69). The normalization can, of course, always be set by an appropriate choice of f.

We must now investigate the restrictions (if any) on the 'quantum number' k entailed by the requirement that the eigenfunction (70) be *univalent* in the plane. The function $\chi(r,\theta)$ is clearly periodic in θ with period $2\pi/|\gamma|$, hence *a fortiori* it is periodic with period 2π, hence it is *univalent* in the plane; and so will be the function $f[\chi(r,\theta),\kappa(r,\theta)]$ provided $f(\chi,\kappa)$ is *univalent*, as we hereafter assume. The function $\varphi(r,\theta)$ is clearly also *univalent*, but this is not generally the case for the function $[\varphi(r,\theta)]^{k/2}$ that appears on the right-hand side of (70*a*), since (for arbitrary k) it features branch points (as function of the *complex* variable $z=r\exp(\mathrm{i}\theta)$) at the equilibrium points η_j, see (62), as well as at the improper equilibrium configurations ($r=0$ respectively $r=\infty$ for $\gamma<-1$ respectively $\gamma>1$). In fact, an easy calculation shows that, as one goes around (clockwise or anticlockwise) one of the equilibrium configurations η_j, see (62), the phase of $\varphi(r,\theta)$ advances by $\pm4\pi$. On the other hand, if one travels along a path that surrounds an improper equilibrium configuration, or equivalently that surrounds all η_j, see (62), the phase of $\varphi(r,\theta)$ advances by $\pm4|\gamma|\pi$.

To determine whether $\psi_k(r,\theta)$, see (70*a*), is *univalent*, we may clearly ignore the first prefactor $r^{(2\gamma-1)/2}$, which is certainly *univalent*. On the other hand, the second prefactor $f[\chi(r,\theta),\kappa(r,\theta)]$ only depends on the orbit. For $\psi(r,\theta)$ to be univalent, it is therefore necessary that, for every orbit on which this prefactor is non-vanishing, the phase advance of $\varphi(r,\theta)^{k/2}$ be an *integer multiple* of 2π. As we have seen above, if the orbit is of class j, with $j\neq0$, the phase advance of $\varphi(r,\theta)$ is $\pm4\pi$. Hence if there are orbits of type j, $j\neq0$, such that $f[\chi(r,\theta),\kappa(r,\theta)]\neq0$, then k must be restricted to be an *integer*. If, on the other hand, *all* orbits on which $f[\chi(r,\theta),\kappa(r,\theta)]\neq0$ are of class *zero*, then the phase advance of $\varphi(r,\theta)$ is $\pm4|\gamma|\pi$, so that k is then only restricted to an *integer multiple of* $1/\gamma$, $k=\tilde{k}/\gamma$ with $\tilde{k}$ *integer*.

This conclusion confirms (as of course it should) the validity of our result: as stated earlier, in each *isochronous* region, the energy spectrum is *equi-spaced*, the eigenvalues coinciding with all *integer* multiples of $\pi\hbar/T_j \equiv h/T_j$, where T_j is the period in the region of configuration space Ω_j. Here we have period $T = 2\pi/\omega$ (see (1)) whenever $\kappa(r,\theta) \neq 0$ and period $|\gamma|T$ otherwise. The above solution shows that the spectrum follows the same rule: for functions whose support includes regions where $\kappa(r,\theta) \neq 0$, the spectrum is given by $\hbar\omega\tilde{k}$; for functions localized *exclusively* in the region characterized by $\kappa(r,\theta) = 0$, the spectrum is given by $\hbar\omega\tilde{k}/\gamma$, where $\tilde{k}$ is of course, in both cases, an *arbitrary integer*.

Let us moreover analyse the eigenfunction (70) by identifying the *classical* significance of each of its constituents. Firstly, let us consider the factor $r^{(2\gamma-1)/2}$ on the right-hand side of (70*a*). It is easy to see that it satisfies the PDE

$$Hr^{(2\gamma-1)/2} = 0, \tag{72}$$

with the differential operator H given by (68). Clearly this factor in the eigenfunction $\psi_k(r,\theta)$ takes care of the presence of the last term on the right-hand side of (68), associated with the symmetrization of the Hamiltonian operator. Secondly, let us consider the functions $\chi(r,\theta)$ and $\kappa(r,\theta)$. As we have seen before, these are both classically conserved quantities and serve to label the orbits. The prefactor $f[\chi(r,\theta),\kappa(r,\theta)]$ therefore plays the same role as the prefactor $\psi(r,\theta_0)$, see (39), in the elementary example treated in section 4.1. Finally, $\varphi(r,\theta)$ is only characterized by its phase (since it has unit modulus) and plays the role of $\exp\left(\frac{\mathrm{i}S}{\hbar}\right)$, see (19).

Finally, at the risk of belabouring the issue (but the purpose of the examples is precisely to display in complete detail the mechanism that underlies our main result), let us outline how the results reported above can be arrived at by following the same steps as in the general reasoning employed in section 3. To this end, we first need to evaluate—in the *classical* context—the time required to reach a point (r,θ) from a given curve. We choose starting points on a set of $|\gamma|$ curves Σ_j, each of them defined as the straight line that goes from the origin to infinity through the jth equilibrium point, and study the wavefunctions generated by starting the orbits on Σ_j. Orbits of class zero intersect, of course, all the curves Σ_j exactly once, whereas orbits of class $j > 0$ intersect one Σ_j twice and do not intersect any other $\Sigma_{j'}$ with $j' \neq j$.

We must now determine the time needed to reach (r,θ) from Σ_j. This is given by $\mathcal{T}[r,\theta;r(0),\theta_j]$, see (67*b*). But in fact there is only one $r(0)$ (or possibly two, see above) from which the trajectory will reach just the coordinates r,θ from Σ_j: let us call it $r^{(j)}(r,\theta)$. It can be determined by requiring the argument of the logarithmic function on the right-hand side of (67*b*) with $\theta_0 = \theta_j$ to be unity: since $\exp(\mathrm{i}\gamma\theta_j) = \mathrm{i}$, it follows that

$$\left|1 - \left(\frac{r^{(j)}}{R}\right)^{\gamma}\right| = \left|1 + \mathrm{i}\left(\frac{r}{R}\right)^{\gamma}\exp(\mathrm{i}\gamma\theta)\right| \tag{73a}$$

entailing

$$\left(\frac{r^{(j)}}{R}\right)^{\gamma}\left[\left(\frac{r^{(j)}}{R}\right)^{\gamma} - 2\right] = \left(\frac{r}{R}\right)^{\gamma}\left[\left(\frac{r}{R}\right)^{\gamma} - 2\sin(\gamma\theta)\right] = \chi(r,\theta) \tag{73b}$$

hence

$$\left(\frac{r^{(j)}}{R}\right)^{\gamma} = 1 \pm [1 + \chi(r,\theta)]^{1/2}. \tag{73c}$$

One readily sees that the right-hand side, $\chi(r,\theta)$, of (73*b*) is always larger than $-\sin^2(\gamma\theta)$, hence the argument of the square root on the right-hand side of the last equation is certainly *nonnegative*, hence both these two solutions, (73*c*), are real and at least on is *positive*; in

fact, both are *positive* iff $\chi(r,\theta)$ is *negative*. The first case corresponds to a classical orbit of class zero. The second case corresponds to a classical orbit of class j with $j \neq 0$. Note incidentally that the above equations determine the initial radius $r^{(j)}$ from the values of r and θ, but not the initial angle θ_j, or in other words, they do not determine around which η_j the orbit going through (r,θ) is turning. This is obtained from more detailed geometrical considerations, which are however unessential, since these questions can all be settled using the $|\gamma|$-fold symmetry of the geometry of the problem.

We therefore now define the following function of r and θ only:

$$t(r,\theta) = \mathcal{T}[r,\theta; r^{(j)}(r,\theta), \theta_j], \tag{74}$$

see (67*b*). It follows immediately from the meaning of $t(r,\theta)$, and it can also be verified algebraically (if need be), that this function satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} t[r(t),\theta(t)] = 1 \tag{75}$$

(when $r(t)$ and $\theta(t)$ evolve according to the equations of motion (58)), and from this there also follows, using again the equations of motion (58), the formula

$$\left\{\left[1 - \left(\frac{r}{R}\right)^{-\gamma} \sin(\gamma\theta)\right] \frac{\partial}{\partial\theta} + \left(\frac{r}{R}\right)^{-\gamma} \cos(\gamma\theta) r \frac{\partial}{\partial r}\right\} t(r,\theta) = 1. \tag{76}$$

An important issue concerns the possible multivaluedness of $\mathcal{T}[r,\theta; r(0),\theta(0)]$ and hence of $t(r,\theta)$. As is readily seen, (67*b*) is in fact a relation between $\exp(-\mathrm{i}\theta)$ and a *univalent* function of r and θ (remembering that γ is an integer). For this reason, $t(r,\theta)$ has no further multivaluedness than an additive constant equal to T times an *integer* (including, as we have seen above, the case in which the change in t after a single round along a closed orbit is $|\gamma|\,T$ rather than T).

We may reason similarly about $r^{(j)}(r,\theta)$: it is clearly conserved by the dynamics, so that

$$\left\{\left[1 - \left(\frac{r}{R}\right)^{-\gamma} \sin(\gamma\theta)\right] \frac{\partial}{\partial\theta} + \left(\frac{r}{R}\right)^{-\gamma} \cos(\gamma\theta) r \frac{\partial}{\partial r}\right\} r^{(j)}(r,\theta) = 0. \tag{77}$$

Bringing all this together, and taking moreover into account the symmetrization of the Hamiltonian, one is clearly led to the eigenfunction (70).

4.4. An interesting example

In this section we consider, as fourth and last example, an 'Aristotelian' three-body problem that we deem interesting in view of its connection with a recent discussion of the transition (in a classical context) from regular to irregular motions [3]. However here we restrict consideration to the two-body case, which does not give rise to any such phenomenology (see below), and our presentation is quite terse, amounting essentially to showing that this case is merely a special case of that treated in the preceding section 4.3.

The three-body problem in question is characterized by the Hamiltonian [3]:

$$H(\underline{z},\underline{p}) = -\sum_{n=1}^{3} \left[\mathrm{i}\omega z_n p_n + g_n \frac{p_{n+1} - p_{n+2}}{z_{n+1} - z_{n+2}}\right], \tag{78}$$

hence by the equations of motion

$$\dot{z}_n = -\mathrm{i}\omega z_n + \frac{g_{n+2}}{z_n - z_{n+1}} + \frac{g_{n+1}}{z_n - z_{n+2}}, \qquad n = 1,\ldots,3, \tag{79}$$

where the coordinates z_n characterizing the motion are *complex*.

Here we restrict consideration to the two-body case by setting $g_1 = g_2 = 0, g_3 = g$. Then the motion of the third particle ($n = 3$) is quite trivial (the elementary rotator, as treated in section 4.1 both in the classical and the quantal contexts), and we can ignore it. We then consider the motion of the other two 'particles', with labels 1 and 2, and we separate the motion of their centre-of-mass ($Z = (z_1 + z_2)/2$), which consists again only of a rotation, and is therefore also rather trivial both in the classical and quantum contexts (as described in section 4.1). We therefore only focus on the *relative* motion of the two interacting particles, as described by the *relative* coordinate $z = z_1 - z_2$ and momentum $p = (p_1 - p_2)/2$, and we reinterpret motions in the complex z-plane as motions in a real plane, by replacing this complex coordinate by the circular coordinates via the standard identification $z(t) = r(t)\exp[\mathrm{i}\theta(t)]$. It is then a matter of trivial algebra—and of taking advantage of the standard transition from a *complex* Hamiltonian problem to an equivalent *real* Hamiltonian problem, see for instance [5]—to find out that our problem turns out to be characterized by the Hamiltonian

$$H(r, \theta; p_r, p_\theta) = -\frac{\omega}{2}\left\{\left[1 - \left(\frac{R}{r}\right)^2 \sin(2\theta)\right] p_\theta + \left(\frac{R}{r}\right)^2 \cos(2\theta) r p_r\right\}, \tag{80a}$$

where we set

$$g = \frac{\omega R^2}{4}. \tag{80b}$$

The final observation is that this Hamiltonian coincides with the special case with $\gamma = 2$ of the Hamiltonian treated in the preceding section 4.3.

5. Outlook

As is generally the case, after our main finding has been internalized, it becomes rather obvious. Indeed it originates from the fact that Hamiltonians that are *linear* in the momenta yield a *quantum-mechanical* behaviour that is quite closely related to their *classical* behaviour. The mathematical underpinning of this fact is that quantum mechanics with Hamiltonians *linear* in the momenta leads to *first-order* PDEs which can then be solved along *characteristics* which essentially coincide with the *classical trajectories*. This implies that these models do *not* display, in the quantum mechanical context, the phenomenon of wavefunction diffusion which in more general systems causes the quantum mechanical model to 'feel' always the entire available phase space: it is in contrast possible to obtain eigenfunctions of these Hamiltonians entirely localized on any (open) set of periodic orbits. It is for this reason that, in the quantum context, *equi-spaced* spectra emerge even for Hamiltonian models which, in the classical context, behave *isochronously* only in certain parts of their natural phase space, but possibly in a very complicated manner (even chaotically) in other parts of their natural phase space. For instance, in the example of section 4.3, we have seen how two spectra, both equi-spaced but with a different spacing, could be associated with two different parts of configuration space. Such attribution of wavefunctions to phase space cannot be performed rigorously in the general case. Nevertheless, attempts have been made: thus Berry and Robnik [6] have discussed (in the semiclassical limit) the spectra of mixed chaotic and integrable systems in terms of a superposition of an integrable and a chaotic spectrum.

So Hamiltonians that are *linear* in the momenta are quite peculiar. Yet they may well yield interesting time evolutions in the classical context, as shown by the example recently introduced to illustrate a new paradigm for the transition from regular to irregular motions [3]. This example provided an important motivation for the study reported in this paper. We treated in the preceding section 4.4 this model, but only in the two-body case; the application

of our findings to the three-body case is likely to be far from trivial, and it warrants additional study.

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Appendix. Derivation of the Hamiltonian (56)

In this appendix we indicate how the Hamiltonian (56) is arrived at, and how the solution (59) of its (classical) equations of motion is obtained. Here the treatment is restricted to the classical case.

Our starting point is the Hamiltonian

$$\tilde{H}(z;\, p) = \frac{\omega}{\gamma}\left[-\mathrm{i} + \left(\frac{z}{R}\right)^{-\gamma}\right] zp, \tag{A.1}$$

where we indicate with z the canonical variable and with p the corresponding canonical momentum. Note that this is now a *one-degree of freedom* problem, but—as indicated by the presence of the imaginary unit i—we are now working with *complex* numbers z, p (we however assume both constants, ω and R, to be *real*, indeed for definiteness *positive*).

The (first) Hamiltonian equation of motion yielded by this model reads

$$\dot{z} = \frac{\omega}{\gamma}\left[-\mathrm{i} + \left(\frac{z}{R}\right)^{-\gamma}\right] z. \tag{A.2}$$

It is then easily seen that, by defining $r(t)$ respectively $\theta(t)$ as the modulus respectively the phase of the complex number $z(t)$,

$$z(t) = r(t)\exp[\mathrm{i}\theta(t)], \tag{A.3}$$

the (*complex*) ODE (A.2) implies that $r(t)$ and $\theta(t)$ evolve precisely according to the (*real*) Hamiltonian equations (58). And indeed the relation among the (*complex*) Hamiltonian $\tilde{H}$ and the (*real*) Hamiltonian H, see (56), is, as can be easily verified,

$$H(x,\, y;\, p_x,\, p_y) = \mathrm{Re}[\tilde{H}(x + \mathrm{i}y;\, p_x - \mathrm{i}p_y)], \tag{A.4}$$

where x, y are the Cartesian coordinates associated with our point particle moving in the (xy)-plane according to the *real* Hamiltonian H. These Cartesian coordinates and momenta are of course related to the circular coordinates and momenta by the standard relations (32).

Hence, rather than solving (58) (apparently a nontrivial task), we can solve (A.2) and then use (A.3). Solving (A.2) is no problem, but we now indicate an elegant ('tricky') way to do it that provides also a hint of how the (*isochronous*) Hamiltonian (A.1) was invented to begin with. Let

$$z(t) = \exp(\mathrm{i}\lambda\omega t)\zeta(\tau), \qquad \tau = \frac{\exp(\mathrm{i}\lambda t) - 1}{\mathrm{i}\omega}. \tag{A.5}$$

This change of dependent and independent variables contains the number λ which we reserve to choose at our convenience, see below. It clearly entails the following simple relation among the initial values of the new and old dependent variables,

$$z(0) = \zeta(0), \tag{A.6}$$

and moreover that, as can be easily verified, the new dependent variable $\zeta(\tau)$ satisfies the following ODE (where of course appended primes denote differentiations with respect to the new independent variable τ):

$$\zeta' = \frac{\omega}{\gamma}\zeta\left[(\gamma\lambda+1)\mathrm{i}(1+\mathrm{i}\omega\tau)^{-(\lambda+1)/\lambda} - \left(\frac{\zeta}{R}\right)^{-\gamma}(1+\mathrm{i}\omega\tau)^{(\gamma\lambda-1)/\lambda}\right]. \tag{A.7}$$

We now make the convenient assignment

$$\lambda = -\frac{1}{\gamma}, \tag{A.8}$$

whereby this ODE, (A.7), takes the following simple (*autonomous*) form:

$$\zeta' = -\frac{\omega}{\gamma}R^{\gamma}\zeta^{1-\gamma}. \tag{A.9}$$

Solving this ODE is now a really trivial task, and via (A.6) and (A.5) with (A.8) it yields the solution (59).

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